

Freudenthal Dual Lagrangians

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ABSTRACT

Freudenthal duality (F-duality) was originally introduced as a non-polynomial anti-involution acting on the electromagnetic charges carried by extremal black hole solutions in the broad class of supergravity theories with U-duality groups of type E_7 . In this context F-duality is a symmetry of the Bekenstein-Hawking entropy. Here it is shown that, by adopting the doubled Lagrangian formalism, F-duality, defined on the doubled field strengths, is not only a symmetry of the black hole solutions, but also of the equations of motion themselves. A further role for F-duality is introduced in the context of world-sheet actions. The Nambu-Goto world-sheet action in any (t, s) signature spacetime can be written in terms of the F-dual. The corresponding field equations and Bianchi identities are then related by F-duality allowing for an F-dual formulation of Gaillard-Zumino duality on the world-sheet. An equivalent polynomial “Polyakov-type” action is introduced using the so-called *black hole potential*. Such a construction allows for actions invariant under all groups of type E_7 , including E_7 itself, although in this case the stringy interpretation is less clear.

1 Introduction

Freudenthal duality (\tilde{F} -duality) was originally conceived as a non-polynomial transformation on the electromagnetic charges x of extremal single-centre large black hole solutions in $D = 4$ supergravities with U-duality group of type E_7 [1]. It is shown here that the generalised scalar-dependent \hat{F} -duality introduced in [2] is in fact a symmetry of the equations of motion and is not restricted to the extremal black hole solutions. This result holds for any ($\mathcal{N} \geq 2$, $D = 4$) symplectic geometry and thus goes beyond supergravities with U-duality group of type E_7 .

The concept of F-duality is also applied here to world-sheet actions. The Nambu-Goto world-sheet action in any (t, s) signature spacetime can be written in terms of the \tilde{F} -dual, making \tilde{F} -duality a manifest symmetry of the action. The corresponding field equations and Bianchi identities are then related by \tilde{F} -duality allowing for an \tilde{F} -dual formulation of Gaillard-Zumino duality [3–5] on the world-sheet. Borrowing ideas from black hole physics in supergravity an equivalent quadratic “Polyakov-type” action is introduced using the black hole potential. Such a construction allows for actions invariant under all groups of type E_7 , including E_7 itself, although in this case the stringy interpretation is less clear.

Before going into more details let us briefly review the key aspects of F-duality as they stand in the current literature [1, 2, 6].

1.1 Background: F-dual Black Holes

The Freudenthal triple system (FTS) was introduced in 1954 by Hans Freudenthal in order to understand the Lie group E_7 , and its non-trivial minuscule representation, in terms of the exceptional simple Jordan algebra [7, 8]. This structure is elegantly captured by the axiomatization of the FTS [9, 10], described below, which may be regarded as *defining* a class of groups sharing a common structure, as encapsulated in the axioms. Such groups are said to be “of type E_7 ”. The essential properties of F-duality follow from the algebraic structure underlying the groups of type E_7 . From a more physical perspective these features can be traced back to the generalised special geometry underpinning $\mathcal{N} \geq 2$ supergravity, which subsumes those theories of type E_7 [2].

Axiomatically, a Freudenthal triple system is a vector space \mathfrak{F} , defined over a field¹ \mathbb{F} not of characteristic 2 or 3, equipped with three maps: (i) a non-degenerate bilinear antisymmetric form $\{x, y\} = -\{y, x\} \in \mathbb{F}$, (ii) a not identically zero four-linear quartic norm $\Delta(x, y, z, w) \in \mathbb{F}$, (iii) a trilinear triple product $T(x, y, z) \in \mathfrak{F}$ defined² by $\{T(x, y, z), w\} = 2\Delta(x, y, z, w)$, where $x, y, z, w \in \mathfrak{F}$. These are then required to satisfy the defining FTS relation,

$$3\{T(x, x, y), T(y, y, y)\} = 2\{x, y\}\Delta(x, y, y, y). \quad (1.1)$$

The *automorphism* group $\text{Aut}(\mathfrak{F})$, defined as the set of invertible \mathbb{F} -linear transformations preserving the quartic norm Δ and the symplectic form $\{x, y\}$, defines a group of type E_7 .

The original black hole \tilde{F} -dual is defined on an *integral* FTS [1, 11], an integral structure denoted $\mathfrak{F}_{\mathbb{Z}}$ corresponding to $\mathfrak{F}_{\mathbb{R}}$, which breaks the automorphism group to a discrete subgroup $\text{Aut}(\mathfrak{F}_{\mathbb{Z}})$, e.g. $E_{7(7)}(\mathbb{Z})$ in the case of $\mathcal{N} = 8$ [12]. It has two particularly interesting features: (a) it is an anti-involution $\tilde{\tilde{x}} = -x$, (b) it leaves invariant the leading-order Bekenstein-Hawking entropy. Moreover, in the maximally supersymmetric $\mathcal{N} = 8$ case there is evidence, deriving from an $E_{7(7)}(\mathbb{Z})$ -invariant dyon degeneracy formula valid on a subset of 1/8-BPS states [13], that the entropy is in fact invariant under \tilde{F} -duality to all orders [1].

Recently, groups of type E_7 , FTS (with $\mathbb{F} = \mathbb{R}, \mathbb{C}$), and F-duality have also appeared in several related, but not directly, contexts. They have been investigated in relation to minimal coupling of vectors and scalars in cosmology and supergravity [14, 15]. F-duality also recently appeared in *Freudenthal*

¹When needed we will denote an FTS defined over a specific field \mathbb{F} by $\mathfrak{F}_{\mathbb{F}}$.

²For convenience we have rescaled the usual definition of the quartic norm $q(x, y, z, w) = 2\Delta(x, y, z, w)$, where $\{T(x, y, z), w\} = q(x, y, z, w)$.

gauge theory, where the scalar fields are $\mathfrak{F}_{\mathbb{R}}$ -valued, giving rise to gauge and global symmetries [6]. Another application is in the context of entanglement in quantum information theory [16–21], with F-duality making a particularly important appearance in [22]. This is actually related to its application to black holes via the black-hole/qubit correspondence [23, 24]. Finally, the most general solution to the supersymmetric stabilization equations in $\mathcal{N} = 8$, $D = 4$ supergravity was expressed in terms of the F-dual of a suitably defined real 56-dimensional vector \mathcal{I} , whose components are real harmonic functions in the Euclidean \mathbb{R}^3 transverse space [25].

Classically ($\mathbb{F} = \mathbb{R}$), the black hole charges x are \mathfrak{F} -valued and transform linearly under the automorphism group $\text{Aut}(\mathfrak{F})$, which, modulo its two-element centre, is isomorphic to the corresponding U-duality group. In this language the black hole F-duality (which we will denote by \tilde{F})

$$\tilde{F} : \mathfrak{F}_0 \rightarrow \mathfrak{F}_0, \quad x \mapsto \tilde{F}(x) =: \tilde{x}, \quad \text{where} \quad \mathfrak{F}_0 := \{x \in \mathfrak{F} \mid \Delta(x, x, x, x) \neq 0\} \quad (1.2)$$

is defined by

$$\tilde{x} := \text{sgn}(\Delta(x)) \frac{T(x)}{\sqrt{|\Delta(x)|}} = \nabla \sqrt{|\Delta(x)|}, \quad (1.3)$$

where $T(x) = T(x, x, x)$ and $\Delta(x) = \Delta(x, x, x, x)$ and we have introduced the gradient operator

$$\nabla := \Omega_{ab} \frac{\partial}{\partial x_b} \quad (1.4)$$

in a basis $\{e^a\}$ using the symplectic metric Ω defined by $\{x, y\}$ as in (2.2). The Bekenstein-Hawking entropy is given by

$$S_{\text{BH}} = \pi \sqrt{|\Delta(x)|} = \pi \sqrt{|\Delta(\tilde{x})|} \quad (1.5)$$

Quantum mechanically the charges x are subject to the Dirac-Schwinger quantisation conditions and so lie on a lattice. Consequently, as previously emphasised, they must be assigned to elements of an *integral* FTS $\mathfrak{F}_{\mathbb{Z}} = \mathfrak{F}(\mathfrak{J}_{\mathbb{Z}})$ where $\mathfrak{J}_{\mathbb{Z}}$ is an *integral* cubic Jordan algebra [1, 11, 26–28]. The corresponding U-duality is given by the discrete automorphism group $\text{Aut}(\mathfrak{F}_{\mathbb{Z}})$, defined as the set of invertible \mathbb{Z} -linear transformations preserving the quartic norm Δ and the symplectic form $\{x, y\}$, which are \mathbb{Z} -valued in this case. This integral structure introduces several subtleties, in particular, not every charge configuration has a well defined \tilde{F} -dual [1]. Moreover, in the classical ($\mathfrak{F}_{\mathbb{R}}$) case every \tilde{F} -dual pair of charge vectors is also U-dual, whereas in the quantum ($\mathfrak{F}_{\mathbb{Z}}$) case this is no longer true. However, these consideration will be largely overlooked here, suffice to say that our results are equally valid in either case, $\text{Aut}(\mathfrak{F}_{\mathbb{R}})$ vs. $\text{Aut}(\mathfrak{F}_{\mathbb{Z}})$. The interested reader can refer to [1, 29, 30] for more on the discrete story.

We will refer to the \tilde{F} -dual as “critical” in the sense that it is defined in terms of only the charges and is independent of the scalar fields. In [2] an interpolating scalar-dependent “non-critical” formulation of the F-duality (which we will denote by \hat{F})

$$\hat{F} : \mathfrak{F}_0 \rightarrow \mathfrak{F}_0, \quad x \mapsto \hat{F}(x) =: \hat{x}, \quad (1.6)$$

was introduced using the so-called effective black hole potential [31] $V_{\text{BH}}(\phi, x)$:

$$\hat{x}(\phi) := \nabla V_{\text{BH}}(\phi, x), \quad \text{where} \quad V_{\text{BH}}(\phi, x) := -\frac{1}{2}\{x, \mathcal{S}(\phi)x\}. \quad (1.7)$$

in the conventional FTS (but unconventional supergravity³) notation.

For U-duality groups of type E_7 $\mathcal{S}(\phi) = \Omega \mathcal{M}(\phi) \in \text{Aut}(\mathfrak{F})$ is a scalar-dependent almost complex structure which may be regarded as the projection onto the adjoint in the symmetric tensor product of the representation carried by \mathfrak{F} . However, since the \hat{F} -dual is defined in terms of the black hole potential only, it is consistent for any ($\mathcal{N} \geq 2$, $D = 4$) symplectic geometry and thus goes beyond supergravities

³Our conventions are explained in detail in section 2.

with U-duality group of type E_7 [2], as will be explained in section 2.3. The generalised \hat{F} -dual is again an anti-involution $\hat{x} = -x$ and, moreover, coincides with the scalar-independent definition when evaluated at the black hole horizon, i.e. at the critical points of the black hole potential,

$$\hat{x}_* = \hat{x}(\phi_*) = \tilde{x}, \quad (1.8)$$

where ϕ_* denotes the horizon value of the moduli. Note, the black hole potential is itself invariant under the generalised \hat{F} -dual [2]

$$V_{\text{BH}}(\phi, \hat{x}) = V_{\text{BH}}(\phi, x), \quad (1.9)$$

which follows, in the case of groups of type E_7 , from $\{\sigma x, \sigma y\} = \{x, y\}, \forall \sigma \in \text{Aut}(\mathfrak{F})$. This implies that while two \hat{F} -dual black holes have generically distinct off-shell effective potentials $V(\phi, x) \neq V(\phi, \tilde{x})$, their entropy is the same despite the fact their (attractor) scalar flows are typically different.

1.2 Beyond Black Hole Solutions

Thus far our discussion of F-duality has been confined to a rather restricted class of black hole solutions, initially focussing on the charges carried by those solutions and then in terms of the effective potential describing their scalar dynamics. This raises the natural question, which we aim to address in the present work: can F-duality be reformulated as a symmetry of the full theory or is it only defined at the level of the extremal black hole solutions?

In response we will show in section 2.5 that, for those theories with U-dualities of type E_7 , the scalar-dependent \hat{F} -dual can be promoted to a symmetry of the theory itself. Since Freudenthal duality relies crucially on the properties of the representation carried by \mathfrak{F} , formulations of the supergravity (specifically its bosonic sector) which make U-duality manifest are the most convenient set-ups with which to address this question. The doubled Lagrangian framework of [32], in which the Lagrangian is written in terms of doubled gauge potentials treated as independent fields but supplemented by a U-duality invariant constraint equation, provides such a construction. We emphasise that the price one must pay in order to write down a simultaneously Lorentz and U-duality invariant Lagrangian is the need for a constraint, which cannot be derived from the Lagrangian and so must be imposed by hand. As such it is essentially defined only at the classical level and its quantisation remains unclear. For our purposes, however, the doubled Lagrangian formalism simply provides a convenient framework to make F-invariance manifest. Symmetries of the doubled Lagrangian and its constraint are classical symmetries of the corresponding supergravity. Defining the \hat{F} -duality operation on the \mathfrak{F} -valued doubled field strengths, it is easily seen to be a symmetry of the doubled Lagrangian and of the constraint itself. Hence, we may conclude that F-duality is a symmetry of the supergravity equations of motion and it is *not* restricted to their black hole solutions⁴.

F-duality is actually a symmetry of another class of theories: Nambu-Goto world-sheet actions in a spacetime with signature (t, s) . In [34] it was shown that the Nambu-Goto action in $(2, 2)$ signature spacetime could be written as the square root of a quartic norm given by Cayley's hyperdeterminant [35], which belongs to an FTS. In this context it was used to illustrate a hidden triality invariance that is specific to $(2, 2)$ signature. However, Cayley's hyperdeterminant is the quartic norm of but one FTS in a countably infinite sequence with automorphism group $\text{SL}(2, \mathbb{R}) \times \text{SO}(t, s)$. These are analogously associated with Nambu-Goto world-sheet actions in (t, s) signature, as is made explicit in section 3. Here, the $\text{SL}(2, \mathbb{R})$ factor corresponds to a global subgroup of the world-sheet diffeomorphisms. The Nambu-Goto actions can then be written in terms of the world-sheet 1-forms F and their Freudenthal duals \tilde{F} ,

$$S_{\text{NG}} = \frac{1}{2} \int d^2 \xi \{ \tilde{F}, F \} \quad (1.10)$$

⁴Recently, in [33] F-duality was enhanced to a continuous and local symmetry acting on the H -variables describing static and extremal black holes of $\mathcal{N} = 2$, $D = 4$ supergravity. However, the resulting Freudenthal gauge symmetry is unrelated to the one constructed in [6].

which makes \tilde{F} -duality manifest. Introducing a set of auxiliary scalar fields, parametrising $[\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(t, s)]/[\mathrm{SO}(2) \times \mathrm{SO}(t, s)]$, and using the idea of the black hole potential we can also define the “non-critical” \hat{F} -dual, $F \mapsto \hat{F}$.

Having described the Nambu-Goto action in (t, s) signature in terms of a specific class of FTS, it is natural to ask whether other FTS could be used in this stringy context. Indeed, recalling that Cartan’s quartic $E_{7(7)}$ invariant I_4 reduces to Cayley’s hyperdeterminant in a canonical basis [36], it was already suggested in [34] that I_4 provides a natural generalisation of the Nambu-Goto action in $(2, 2)$ signature with an $\mathrm{SO}(6, 6)$ Lorentz group embedded in the larger $E_{7(7)}$ symmetry. Here, this idea is developed in section 3.2. In principle one could consider any FTS, however, we focus on the “maximal” case of \mathfrak{F}^{0^s} , where $F \in \mathfrak{F}^{0^s}$ transforms as the fundamental **56** of $\mathrm{Aut}(\mathfrak{F}^{0^s}) = E_{7(7)}$ and $\Delta(F) = I_4(F)$. The notation \mathfrak{F}^{0^s} refers to the fact that it may be constructed using the Jordan algebra of 3×3 Hermitian matrices defined over the split octonions \mathbb{O}^s . This gives a manifestly $E_{7(7)}$ -invariant “world-sheet” action. However, in order to clarify its stringy interpretation, we decompose the **56** w.r.t. $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, 6)$, yielding a world-sheet 1-form F_α^a in a target space with $(6, 6)$ signature coupled to 32 auxiliary world-sheet scalar densities (a target space Weyl spinor) λ^A . These two fields are mixed by the larger global $E_{7(7)}$ symmetry.

1.3 Plan

In section 2 we review the key properties of Freudenthal triple systems and the three known incarnations of Freudenthal duality (i) the “critical” \tilde{F} -duality, which is defined on the black hole charges of supergravities with U-duality groups type E_7 , (ii) the “critical” \check{F} -duality, which is defined on the black hole charges of all $\mathcal{N} \geq 2$ -extended supergravity theories in $D = 4$ using the entropy function, (iii) the scalar-dependent “non-critical” \hat{F} -duality, which is defined on the black hole charges of all $\mathcal{N} \geq 2$ -extended supergravity theories in $D = 4$ using the black hole potential away from its critical points. Furthermore, in section 2.5 we recall the definition of the doubled Lagrangian formalism, and use it to show \hat{F} -duality is an invariance of the equations of motion.

In section 3 we show that the Nambu-Goto action naturally defines an FTS and that it can therefore be written in terms of the \tilde{F} -dual. This yields an \tilde{F} -dual formulation of Gaillard-Zumino duality on the world-sheet. We conclude with some more speculative observations on the existence of world-sheet actions with $\mathrm{Aut}(\mathfrak{F})$ -symmetry for generic \mathfrak{F} , including the exceptional case of E_7 .

2 Freudenthal Duality

2.1 The Freudenthal Triple System

In 1954 Freudenthal [7, 8] found that the 133-dimensional exceptional Lie group E_7 could be understood in terms of the automorphisms of a construction based on the minimal dimensional E_7 -module **56** built from the exceptional Jordan algebra of 3×3 Hermitian octonionic matrices. Today this construction goes by the name of *the Freudenthal triple system*, reflecting the special role played by its triple product.

Following Freudenthal, Meyberg [9] and Brown [10] axiomatized the ternary structure underlying the FTS. The E_7 -module is just one of a class of modules of “groups of type E_7 ”. The FTS carries the representation of the dyonic black hole charge vectors for a broad class of 4-dimensional supergravity theories [37–39].

An FTS is axiomatically defined [10] as a finite dimensional vector space \mathfrak{F} over a field \mathbb{F} (not of characteristic 2 or 3), such that:

1. \mathfrak{F} possesses a non-degenerate antisymmetric bilinear form $\{x, y\}$.
2. \mathfrak{F} possesses a symmetric four-linear form $\Delta(x, y, z, w)$ which is not identically zero.

3. If the ternary product $T(x, y, z)$ is defined on \mathfrak{F} by $\{T(x, y, z), w\} = 2\Delta(x, y, z, w)$, then

$$3\{T(x, x, y), T(y, y, y)\} = 2\{x, y\}\Delta(x, y, y, y). \quad (2.1)$$

It will often be notationally convenient to introduce a basis $\{e^a\}$, $a = 1, \dots, \dim_{\mathbb{F}} \mathfrak{F}$ such that

$$\{e^a, e^b\} = \Omega^{ab}, \quad \Delta(e^a, e^b, e^c, e^d) = K^{abcd}. \quad (2.2)$$

Since $\{x, y\}$ is non-degenerate, we can define the symplectic 2-form Ω_{ab} such that $\Omega_{ab}\Omega^{bc} = \delta_a^c$. We will raise/lower indices with Ω^{ab}/Ω_{ab} .

The *automorphism* group of an FTS is defined as the set of invertible \mathbb{F} -linear transformations preserving the quartic and quadratic forms:

$$\text{Aut}(\mathfrak{F}) := \{\sigma \in \text{Iso}_{\mathbb{F}}(\mathfrak{F}) | \{\sigma x, \sigma y\} = \{x, y\}, q(\sigma x) = q(x)\}. \quad (2.3)$$

Note, the conditions $\{\sigma x, \sigma y\} = \{x, y\}$ and $q(\sigma x) = q(x)$ immediately imply

$$T(\sigma x) = \sigma T(x). \quad (2.4)$$

A remarkable result due to Brown [10] and Ferrar [40] implies that every simple reduced⁵ FTS \mathfrak{F} is isomorphic to an FTS $\mathfrak{F}(\mathfrak{J})$, where

$$\mathfrak{F}(\mathfrak{J}) := \mathbb{F} \oplus \mathbb{F} \oplus \mathfrak{J} \oplus \mathfrak{J} \quad (2.5)$$

and \mathfrak{J} is the Jordan algebra of an admissible cubic form with base point or the Jordan algebra of a non-degenerate quadratic form. The FTS quadratic form, quartic norm and triple product are then defined in terms of the basic Jordan algebra operations [10, 40]; for details, see [41] and Refs. therein. We will not make explicit use of this perspective here, however, it is useful for tabulating the automorphism groups according as the underlying cubic Jordan algebra as we have done in Table 1. From a physical point of view, the underlying Jordan algebra structure originates from the fact that the $D = 4$ supergravities may be obtained by dimensionally reducing a $D = 5$ theory whose U-duality is characterised by the reduced structure group of the corresponding Jordan algebra.

For $\mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}})$ the automorphism group has a two element centre and its quotient yields the simple groups listed in Table 1, while for $\mathfrak{F}(\mathfrak{J}_{m,n})$ one obtains the semi-simple groups $\text{SL}(2, \mathbb{R}) \times \text{SO}(m+1, n+1)$ [10, 11, 42]. In all cases \mathfrak{F} forms a symplectic representation of $\text{Aut}(\mathfrak{F})$, the dimensions of which are listed in the final column of Table 1. This table covers a number 4-dimensional supergravities: $\mathfrak{F}^{2,n} := \mathfrak{F}(\mathfrak{J}_{1,n-1})$ and $\mathfrak{F}^{6,n} := \mathfrak{F}(\mathfrak{J}_{5,n-1})$ respectively correspond to $\mathcal{N} = 2, 4$ Maxwell-Einstein supergravity, while $\mathfrak{F}^{\mathbb{A}} := \mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}})$ correspond to $\mathcal{N} = 2$ “magic” Maxwell-Einstein supergravity, $\mathfrak{F}^{\text{Os}} := \mathfrak{F}(\mathfrak{J}_3^{\text{Os}})$ corresponds to $\mathcal{N} = 8$ maximally supersymmetric supergravity, and $\mathfrak{F}(\mathbb{R})$ corresponds to the $\mathcal{N} = 2$ t^3 model (see, for example, [1, 23, 30, 37, 38, 43–46]).

2.2 Black Hole $\tilde{\text{F}}$ -Duality...

The black hole $\tilde{\text{F}}$ -duality is a non-polynomial transformation on the electromagnetic charges x of extremal single-centre large black hole solutions in $D = 4$ supergravities with U-duality group of type E_7 [1].

The essential characteristics of the $\tilde{\text{F}}$ -dual, defined by (1.2) and (1.3), follow from the defining FTS relation (1.1), which, in particular, implies [10]

$$T(T(x)) = -\Delta^2(x)x. \quad (2.6)$$

⁵An FTS is simple if and only if $\{x, y\}$ is non-degenerate, which we assume. An FTS is said to be reduced if it contains a strictly regular element: $\exists u \in \mathfrak{F}$ such that $T(u, u, u) = 0$ and $u \in \text{Range } L_{u,u}$ where $L_{x,y} : \mathfrak{F} \rightarrow \mathfrak{F}; \quad L_{x,y}(z) := T(x, y, z)$. Note that FTS on “degenerate” groups of type E_7 (as defined in [15], and Refs. therein) are not reduced and hence cannot be written as $\mathfrak{F}(\mathfrak{J})$; they correspond to theories which cannot be uplifted to $D = 5$ dimensions consistently reflecting the lack of an underlying \mathfrak{J} .

Table 1: The automorphism group $\text{Aut}(\mathfrak{F}(\mathfrak{J}))$ and the dimension of its representation $\dim \mathfrak{F}(\mathfrak{J})$ given by the Freudenthal construction defined over the cubic Jordan algebra \mathfrak{J} over \mathbb{R} (with dimension $\dim \mathfrak{J}$ and reduced structure group $\text{Str}_0(\mathfrak{J})$).

Jordan algebra \mathfrak{J}	$\text{Str}_0(\mathfrak{J})$	$\dim \mathfrak{J}$	$\text{Aut}(\mathfrak{F}(\mathfrak{J}))$	$\dim \mathfrak{F}(\mathfrak{J})$
\mathbb{R}	—	1	$\text{SL}(2, \mathbb{R})$	4
$\mathbb{R} \oplus \mathbb{R}$	$\text{SO}(1, 1)$	2	$\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$	6
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$\text{SO}(1, 1) \times \text{SO}(1, 1)$	3	$\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$	8
$\mathbb{R} \oplus \Gamma_{1, n-1}$	$\text{SO}(1, 1) \times \text{SO}(1, n-1)$	$n+1$	$\text{SL}(2, \mathbb{R}) \times \text{SO}(2, n)$	$2(n+2)$
$\mathbb{R} \oplus \Gamma_{5, n-1}$	$\text{SO}(1, 1) \times \text{SO}(5, n-1)$	$n+5$	$\text{SL}(2, \mathbb{R}) \times \text{SO}(6, n)$	$2(n+6)$
$\mathfrak{J}_3^{\mathbb{R}}$	$\text{SL}(3, \mathbb{R})$	6	$\text{Sp}(6, \mathbb{R})$	14
$\mathfrak{J}_3^{\mathbb{C}}$	$\text{SL}(3, \mathbb{C})$	9	$\text{SU}(3, 3)$	20
$\mathfrak{J}_3^{\mathbb{C}^s}$	$\text{SL}(3, \mathbb{R}) \times \text{SL}(3, \mathbb{R})$	9	$\text{SL}(6, \mathbb{R})$	20
$\mathfrak{J}_3^{\mathbb{H}}$	$\text{SU}^*(6)$	15	$\text{SO}^*(12)$	32
$\mathfrak{J}_3^{\mathbb{H}^s}$	$\text{SL}(6, \mathbb{R})$	15	$\text{SO}(6, 6)$	32
$\mathfrak{J}_3^{\mathbb{O}}$	$E_{6(-26)}$	27	$E_{7(-25)}$	56
$\mathfrak{J}_3^{\mathbb{O}^s}$	$E_{6(6)}$	27	$E_{7(7)}$	56

The invariance of $\Delta(x)$ follows by recalling that

$$2\Delta(x) = \{T(x), x\}. \quad (2.7)$$

Hence, using (2.6)

$$\Delta(T(x)) = \Delta(x)^3 \quad (2.8)$$

one finds

$$\Delta(\tilde{x}) = \Delta(T(x))\Delta(x)^{-2} = \Delta(x). \quad (2.9)$$

Moreover, $\tilde{\text{F}}$ -dual is anti-involutive,

$$\tilde{\tilde{x}} = T(\tilde{x})|\Delta(x)|^{-1/2} = T(T(x))\Delta(x)^{-2} = -x. \quad (2.10)$$

Note in particular, that from the above we have

$$\{\tilde{x}, x\} = 2\sqrt{|\Delta(x)|}. \quad (2.11)$$

Since the Bekenstein-Hawking entropy is given by,

$$S_{\text{BH}} = \pi\sqrt{|\Delta(x)|} = \frac{\pi}{2}\{\tilde{x}, x\}, \quad (2.12)$$

it is manifestly invariant under F-duality.

For classical black hole solutions the charges are \mathbb{R} -valued, hence \mathfrak{F} is defined over \mathbb{R} . However, quantum mechanically this is no longer the case. The Dirac-Schwinger quantisation condition relating two black holes with charges x and x' is given by

$$\{x, x'\} \in \mathbb{Z}. \quad (2.13)$$

Consequently, the black hole charges x must be assigned to elements of an *integral* FTS $\mathfrak{F}_{\mathbb{Z}} = \mathfrak{F}(\mathfrak{J}_{\mathbb{Z}})$ where $\mathfrak{J}_{\mathbb{Z}}$ is an *integral* cubic Jordan algebra [11, 26–28]. The corresponding U-duality is given by the discrete automorphism group $\text{Aut}(\mathfrak{F}_{\mathbb{Z}})$, e.g. $E_{7(7)}(\mathbb{Z})$ in the case of $\mathcal{N} = 8$ [12]. In particular, $\Delta(x)$ is now quantised:

$$\Delta(x) \in \{0, 1\} \pmod{4}. \quad (2.14)$$

This integral structure constrains the class of black holes admitting a well-defined \tilde{F} -dual. Requiring that \tilde{x} be integral restricts us to that subset of black holes for which $|\Delta(x)|$ is a perfect square and for which $|\Delta(x)|^{1/2}$ divides $T(x)$:

$$d_4(x) = \left[\frac{d_3(x)}{d_1(\tilde{x})} \right]^2, \quad (2.15)$$

where we have introduced a set of discrete U-duality invariants constructed in terms of the greatest common divisor (gcd) [1, 11, 13]:

$$\begin{aligned} d_1(x) &= \gcd(x) \\ d_2(x) &= \gcd(3T(x, x, y) + \{x, y\}x) \quad \forall y \\ d_3(x) &= \gcd(T(x, x, x)) \\ d_4(x) &= |\Delta(x)| \\ d'_4(x) &= \gcd(x \wedge T(x)). \end{aligned} \quad (2.16)$$

In the classical theory, where the black hole charges are real-valued, any pair of \tilde{F} -dual charge vectors are related by U-duality. However, in the quantum theory it is no longer true that any well defined pair of \tilde{F} -dual black holes are U-dual [1]. This is most directly confirmed by considering the discrete U-duality invariants listed in (2.16). While $d_2(x)$, $d_4(x)$, and $d'_4(x)$ are \tilde{F} -dual invariant $d_1 = \gcd(x)$ and $d_3 = \gcd(T(x))$ need not be. Since corrections to the leading-order Bekenstein-Hawking entropy may depend on the discrete invariants [13, 29, 47–53] it is not clear that \tilde{F} -duality is a symmetry to all orders. However, for a subclass, closed under $E_{7(7)}(\mathbb{Z})$ and F-duality, of 1/8-BPS dyons in type II string theory on a 6-torus Sen [13] derived degeneracy formula which is a function of only d_4 and d'_4 and therefore is \tilde{F} -dual invariant.

2.3 ...and its Generalization Beyond Groups of Type E_7 : \check{F} -Duality

The symplectic structure of $\mathcal{N} \geq 2$ -extended supergravity theories in $D = 4$ spacetime dimensions [54, 55] allows one to generalize the “critical” \tilde{F} -duality *beyond* U-duality groups of type E_7 ; we will here denote such a generalization \check{F} -duality. Its action on $Sp(2n, \mathbb{R})$ charge vectors x is defined as [2] (recall definition (1.4))

$$\check{F} : x \mapsto \check{F}(x) := \check{x} := \nabla S(x) = -\Omega \mathcal{M}(\phi_*(x)) x. \quad (2.17)$$

By indicating with ϕ the set of scalar fields, \mathcal{M} denotes the scalar-dependent, symplectic, negative definite, real symmetric matrix made of the vector couplings [54, 56]:

$$\mathcal{M}(\phi)\Omega\mathcal{M}(\phi) = \Omega, \quad \mathcal{M}^T(\phi) = \mathcal{M}(\phi). \quad (2.18)$$

Furthermore, $S(x)$ denotes the Bekenstein-Hawking black hole entropy (homogeneous of degree two in the charges x):

$$S(x) = \pi V_{\text{BH}}(\phi_*(x), x), \quad (2.19)$$

where $\phi_*(x)$ denotes the x -dependent critical values of the scalar fields, defined as

$$\partial_\phi V_{\text{BH}}(\phi, x)|_{\phi_*(x)} = 0. \quad (2.20)$$

It can be proved [2] that the critical points $\phi_*(x)$ of $V_{\text{BH}}(\phi, x)$ (defined by (2.20)) coincide with the critical points $\phi_*(\check{x})$ of $\check{F}(V_{\text{BH}}(\phi, x)) := V_{\text{BH}}(\phi, \check{x})$ (defined by (2.20) with $x \rightarrow \check{x}$). It thus follows that the symplecticity of \mathcal{M} (2.18) (in particular, evaluated at $\phi = \phi_*(x) = \phi_*(\check{x})$), implies the anti-involutivity of \tilde{F} -duality (2.17):

$$\check{F}^2 : x \mapsto \check{F}^2(x) := \check{F}(\check{F}(x)) = -\Omega \mathcal{M}(\phi_*(\check{x})) \check{x} = \Omega \mathcal{M}(\phi_*(x)) \Omega \mathcal{M}(\phi_*(x)) x = -x. \quad (2.21)$$

An important consequence of these results is that the Bekenstein-Hawking entropy $S(x)$, which generally is a complicated non-polynomial function homogenous of degree two in the charges x , is \check{F} -invariant [2]:

$$S(x) = \pi V_{\text{BH}}(\phi_*(x), x) = \pi V_{\text{BH}}(\phi_*(\check{x}), \check{x}) = S(\check{x}) \Leftrightarrow S(x) = S(\nabla S(x)). \quad (2.22)$$

(2.22) is a general result, which holds in any ($\mathcal{N} \geq 2$, $d = 4$) generalized special geometry [2, 55]. Within this broad class of theories, two \check{F} -dual black holes (namely, two black holes whose dyonic charge vectors x are related by \check{F} -duality (2.17)) have generically different effective potentials $V_{\text{BH}}(\phi, x) \neq V_{\text{BH}}(\phi, \check{x})$, which however exhibit the same critical points ($\phi_*(x) = \phi_*(\check{x})$); thus, as yielded by (2.22), two \check{F} -dual black holes have the same Bekenstein-Hawking entropy, despite the fact their (attractor) scalar flows are generally different.

For those generalized special geometries [55] related to groups of type E_7 [10], \check{F} -duality (2.17) consistently reduces to the non-polynomial (homogeneity-preserving) \hat{F} -duality defined by (1.2)-(1.3). In particular, $S(x)$ reduces to the (π times) the square root of the (absolute value of) the quartic polynomial $\Delta(x)$:

$$\text{groups of type } E_7 : \check{F} = \hat{F} \Rightarrow S(x) = \pi \sqrt{|\Delta(x)|}. \quad (2.23)$$

2.4 Generalised Scalar-Dependent Freudenthal Duality \hat{F}

The scalar-dependent, “non-critical ” F -duality (denoted by \hat{F}) was introduced in [2], and it is defined by (1.6) and (1.7), with

$$\hat{F}(x) =: \hat{x}(\phi) := \partial_x V_{\text{BH}}(\phi, x) = -\mathcal{S}(\phi)x, \quad \text{where} \quad \mathcal{S}(\phi) := \Omega \mathcal{M}(\phi). \quad (2.24)$$

Note that the scalar fields ϕ do not transform under \hat{F} -duality. From its very definition, the real matrix $\mathcal{S}(\phi) \in \text{Aut}(\mathfrak{F})$ is a scalar-dependent almost complex structure which may be regarded as the projection onto the adjoint in the symmetric tensor product of the representation carried by \mathfrak{F} :

$$\mathcal{S}^2 = -\mathbb{I}. \quad (2.25)$$

As a consequence of (2.18), it is immediate to see that \hat{F} is anti-involutive:

$$\hat{\hat{x}}(\phi) = \Omega \mathcal{M}(\phi) \Omega \mathcal{M}(\phi) x = -x. \quad (2.26)$$

Furthermore, \hat{F} -duality coincides with \check{F} -duality when evaluated at the black hole horizon

$$\hat{x}_* = \hat{x}(\phi_*) = \check{F}(x), \quad (2.27)$$

where ϕ_* denotes the horizon value of the scalar fields, defined by (2.20).

As given by (1.9) [2], the black hole potential V_{BH} is itself invariant under the generalised \hat{F} -dual; this result is general: it holds in any ($\mathcal{N} \geq 2$, $d = 4$) symplectic geometry [55].

It can also be checked that all definitions and results concerning \hat{F} -duality consistently reduce, when evaluated at the (non-degenerate) critical points of V_{BH} , to the analogous definitions and results for \check{F} -duality, introduced in section 2.3.

2.5 \hat{F} -Duality in Supergravity

The doubled Lagrangian

$$\mathcal{L}_{\text{double}} = R \star 1 + \frac{1}{4} \text{tr}(d\mathcal{M}^{-1} \wedge \star d\mathcal{M}) - \frac{1}{4} H \wedge \mathcal{M} \star H \quad (2.28)$$

and the twisted self-duality constraint

$$H = \Omega \mathcal{M} \star H \quad (2.29)$$

are equivalent at the level of equations of motion to the standard formulation of supergravity when scalars parametrise a homogeneous symmetric space G_4/H_4 [32]. For the vector equations of motion, the Lagrangian $\mathcal{L}_{\text{double}}$ should be varied with respect to the doubled gauge potentials (A, B) , where $H = d(A, B)$, treated as *independent* fields; the constraint (2.29) is then applied to the equations of motion. Similarly, for the scalar equations of motion one first varies treating $H = (dA, dB)$ as independent and then applies the constraint (2.29).

The “non-critical ” $\hat{\text{F}}$ -duality introduced in section 2.4 may be extended to the doubled field strengths as follows (*cfr.* (2.24)):

$$\hat{H} := -\mathcal{S}(\phi)H, \quad (2.30)$$

such that the constraint (2.29) can be rewritten as

$$H = -\star \hat{H}. \quad (2.31)$$

Since $\mathcal{M} = -\Omega\mathcal{S}$ and $\mathcal{S}^2 = -\mathbb{1}$, it is clear that $\hat{\text{F}}$ -duality defined on the doubled field strengths leaves $\mathcal{L}_{\text{double}}$ invariant (note that $\hat{\text{F}}$ -duality is inert on the Einstein-Hilbert and non-linear sigma model terms). It is also clear that

$$\hat{H} = \Omega\mathcal{M} \star \hat{H} \quad \Leftrightarrow \quad H = \Omega\mathcal{M} \star H, \quad (2.32)$$

and hence $\hat{\text{F}}$ -duality is a symmetry of the full theory, not just of the black hole solutions.

3 Freudenthal-Dual World-Sheet Actions

3.1 The Nambu-Goto Action

We start by considering the Nambu-Goto world-sheet action in a pseudo-Euclidean spacetime with signature (t, s) ,

$$S_{\text{NG}} = \int d^2\xi \sqrt{\det \partial_\alpha X^a \partial_\beta X^b \eta_{ab}}. \quad (3.1)$$

where we have gone to tangent-space spacetime indices. We make the simple observation that \mathcal{L} is the square root of a homogeneous quartic polynomial. In the case of a spacetime with signature $(2, 2)$, it was shown in [57] that \mathcal{L}_{NG} is given by the square root of Cayley’s hyperdeterminant, a quartic invariant of $[\text{SL}(2, \mathbb{R})]^3$, which is the quartic norm of the FTS $\mathfrak{F}(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})$ appearing in the third row of Table 1. Linearizing the quartic polynomial appearing in (3.1) we can write more generally

$$\mathcal{L} = \sqrt{|\Delta(\partial_\alpha X^a, \partial_\alpha X^a, \partial_\alpha X^a, \partial_\alpha X^a)|} \quad (3.2)$$

where Δ is a totally symmetric quartic form defined by

$$\Delta(X_{\alpha_1}^{a_1}, Y_{\alpha_2}^{a_2}, Z_{\alpha_3}^{a_3}, W_{\alpha_4}^{a_4}) = \frac{1}{12} [\epsilon^{\alpha_1\alpha_3} \epsilon^{\alpha_2\alpha_4} \eta_{a_1a_2} \eta_{a_3a_4} + 5 (\alpha_i a_i) \text{ pairwise perms.}] X_{\alpha_1}^{a_1} Y_{\alpha_2}^{a_2} Z_{\alpha_3}^{a_3} W_{\alpha_4}^{a_4}. \quad (3.3)$$

The $6 = 4!/4$ distinct terms are generated by permutations modulo those of the form $(ij)(kl)$ under which $\epsilon^{\alpha_i\alpha_j} \epsilon^{\alpha_k\alpha_l} \eta_{a_i a_j} \eta_{a_k a_l}$ is invariant.

Taking this as the definition of a quartic norm allows us to define the Nambu-Goto Freudenthal triple system \mathfrak{F}_{NG} by introducing a carefully chosen anti-symmetric bilinear form. This pair then defines the triple product which must be checked to satisfy the basic FTS identity (2.1).

Let us pick a basis for the string world-sheet derivatives as a vector space,

$$F = F_\alpha^a e^\alpha \otimes e_a, \quad \text{where} \quad F_\alpha^a = \partial_\alpha X^a, \quad (3.4)$$

and define,

1. The non-degenerate antisymmetric bilinear form (*cfr.* [58, 59])

$$\{e^\alpha \otimes e_a, e^\beta \otimes e_b\} = \epsilon^{\alpha\beta} \eta_{ab} = \Omega. \quad (3.5)$$

2. The not identically zero symmetric quartic norm [59] as determined by the Nambu-Goto Lagrangian,

$$\Delta(e^\alpha \otimes e_a, e^\beta \otimes e_b, e^\gamma \otimes e_c, e^\delta \otimes e_d) = \frac{1}{12} \left[\epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \eta_{ab} \eta_{cd} + 5 \text{ perms.} \right] = K. \quad (3.6)$$

Defining the triple product through

$$\{T(e^\alpha \otimes e_a, e^\beta \otimes e_b, e^\gamma \otimes e_c), e^\delta \otimes e_d\} = 2\Delta(e^\alpha \otimes e_a, e^\beta \otimes e_b, e^\gamma \otimes e_c, e^\delta \otimes e_d) \quad (3.7)$$

one finds

$$T(e^\alpha \otimes e_a, e^\beta \otimes e_b, e^\gamma \otimes e_c) = T_{abc\delta}^{\alpha\beta\gamma d} e^\delta \otimes e_d \quad (3.8)$$

where

$$T_{abc\delta}^{\alpha\beta\gamma d} = K_{abcd'}^{\alpha\beta\gamma\delta'} \epsilon_{\delta'\delta} \eta^{d'd} = \frac{1}{12} \left[\epsilon^{\alpha\gamma} \delta_\delta^\beta \eta_{ab} \delta_c^d + 5 \text{ perms.} \right]. \quad (3.9)$$

Using,

$$K_{abcd}^{\alpha\beta\gamma\delta} T_{a'b'c'd}^{\alpha'\beta'\gamma'\delta} = \frac{1}{12} \left[K_{abcc'}^{\alpha\beta\gamma\beta'} \epsilon^{\alpha'\gamma'} \eta_{a'b'} + 5 \text{ perms.} \right] \quad (3.10)$$

one obtains

$$3\{T(X, X, Y), T(Y, Y, Y)\} = \frac{1}{6} \left[K_{abcc'}^{\alpha\beta\gamma\beta'} \epsilon^{\alpha'\gamma'} \eta_{a'b'} + 5 \text{ perms.} \right] X_\alpha^a X_\beta^b Y_\gamma^c Y_{\alpha'}^{a'} Y_{\beta'}^{b'} Y_{\gamma'}^{c'} \quad (3.11)$$

which, using (3.6), gives

$$3\{T(X, X, Y), T(Y, Y, Y)\} = 2\{X, Y\} \Delta(X, Y, Y, Y) \quad (3.12)$$

as required by the third postulate defining a Freudenthal triple system (*cfr.* (2.1)). Hence, the $2(s+t)$ -dimensional vector space spanned by $\{e^\alpha \otimes e_a\}$ forms a Freudenthal triple system \mathfrak{F}_{NG} . The automorphism group is given by $\text{SL}(2, \mathbb{R}) \times \text{SO}(t, s)$ clearly corresponding to the Freudenthal triple system over the semi-simple rank-3 Jordan algebra $\mathbb{R} \oplus \mathbf{\Gamma}_{s-1, t-1}$ (of which the fourth and fifth rows in Table 1 are particular cases, see [41] for details), namely:

$$\mathfrak{F}_{\text{NG}} = \mathfrak{F}(\mathbb{R} \oplus \mathbf{\Gamma}_{t-1, s-1}). \quad (3.13)$$

In this context the $\text{SL}(2, \mathbb{R})$ factor is just a global subgroup of the world-sheet diffeomorphisms.

Note, this FTS has previously appeared in physics literature [60, 61]. In particular, quasiconformal realisations over these FTS were used to capture the three dimensional U-duality groups as spectrum generating quasiconformal groups in [61].

Consequently, the Nambu-Goto world-sheet action may be written using the $\tilde{\text{F}}$ -dual,

$$S_{\text{NG}} = \frac{1}{2} \int d^2 \xi \{ \tilde{F}, F \} = \int d^2 \xi \sqrt{|\Delta(F)|} \quad (3.14)$$

It is now manifest that the Nambu-Goto action is in fact invariant under $\tilde{\text{F}}$ -duality since,

$$\{\tilde{F}, F\} \mapsto \{-F, \tilde{F}\} = \{\tilde{F}, F\}. \quad (3.15)$$

The equations of motion and Bianchi identities are then simply,

$$d \star \begin{pmatrix} \tilde{F} \\ F \end{pmatrix} = 0 \quad (3.16)$$

where \star denotes the world-sheet Hodge dual. Hence the equations of motion and Bianchi identities are interchanged by \hat{F} -duality.

Using the idea of the black hole potential we can introduce an equivalent Polyakov-type action of the form,

$$S_{\text{pot}} = \frac{1}{2} \int d^2\xi \{F, \mathcal{S}(\Phi)F\} = -\frac{1}{2} \int d^2\xi F \mathcal{M}(\Phi) F = \int d^2\xi V_{\text{BH}}(\Phi, F). \quad (3.17)$$

Here $\mathcal{S}(\Phi) = \Omega \mathcal{M}(\Phi) = \epsilon^{\alpha\beta} M_{\beta\delta}(\varphi) \eta^{ab} M_{bc}(\phi) \in \text{SL}(2, \mathbb{R}) \times \text{SO}(t, s)$ is a function of the auxiliary scalar fields $\Phi := (\varphi, \phi)$, which parametrise the coset,

$$\frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)} \times \frac{\text{SO}(t, s)}{\text{SO}(t) \times \text{SO}(s)}, \quad (3.18)$$

and satisfies $\mathcal{S}(\Phi)^2 = -\mathbf{1}$, $\mathcal{M}(\Phi)^T = \mathcal{M}(\Phi)$.

Then, one can then introduce the “non-critical” world-sheet \hat{F} -duality as follows:

$$\hat{F} := -\mathcal{S}(\Phi)F, \quad \hat{\hat{F}} = -F. \quad (3.19)$$

Recalling (3.19), S_{pot} can be rewritten as

$$S_{\text{pot}} = \frac{1}{2} \int d^2\xi \{\hat{F}, F\}, \quad (3.20)$$

which makes \hat{F} -invariance manifest (*cfr.* section 2.4).

Let, $G = \hat{F}$ so that

$$d \star \begin{pmatrix} F \\ G \end{pmatrix} = 0. \quad (3.21)$$

This system of equations is invariant under $\text{GL}(2(s+t), \mathbb{R})$, however one must also preserve the definition of G and the invariance of the equations of motion of any other fields present in the theory. As shown in [4], by following the analysis of Gaillard and Zumino [3], in $D = 2$, and more generally in $D = 4k+2$, the maximal consistent duality group for n $D/2$ -form field strengths is $\text{SO}(n, n)$. It is immediately apparent that the same is true for (3.21) if we treat F, G independently, as they would be in the presence of other interacting fields, but where the spacetime indices play the role of n .

Note that by substituting the equations of motion for the $2 + st$ auxiliary scalar fields Φ back into S_{pot} (3.20), one obtains S_{NG} (3.14). These equations of motion are nothing but the criticality conditions for $V_{\text{BH}}(\Phi, F)$. Indeed, as discussed in section 2, for groups “of type E_7 ” the “non-critical” \hat{F} -duality reduces to the “critical” \tilde{F} -duality, when evaluated at the critical points of V_{BH} .

3.2 Beyond Nambu-Goto

The Nambu-Goto action in (t, s) signature has been described in terms of a specific class of FTS,

$$F = \partial_\alpha X^a e^\alpha \otimes e_a \in \mathfrak{F}_{\text{NG}} = \mathfrak{F}(\mathbb{R} \oplus \mathbf{\Gamma}_{t-1, s-1}), \quad (3.22)$$

where

$$\int d^2\xi \sqrt{|\Delta(F)|} = \int d^2\xi \sqrt{\det \partial_\alpha X^a \partial_\beta X^b \eta_{ab}}. \quad (3.23)$$

In this case the automorphism group, $\text{Aut}(\mathfrak{F}_{\text{NG}}) = \text{SL}(2, \mathbb{R}) \times \text{SO}(t, s)$, has two factors which are naturally identified with the global subgroup of world-sheet diffeomorphisms and the spacetime Lorentz group, respectively.

However, formally we are free to consider the “world-sheet” action

$$S_{\mathfrak{F}} = \int d^2\xi \sqrt{|\Delta(F(\tau, \sigma))|}, \quad (3.24)$$

for an arbitrary FTS, where \mathfrak{F} is an irreducible $\text{Aut}(\mathfrak{F})$ -module. In fact, recalling that Cartan's quartic $E_{7(7)}$ invariant I_4 reduces to Cayley's hyperdeterminant in a canonical basis [36], it was already suggested in [34] that I_4 provides a natural generalisation of the Nambu-Goto action in $(2, 2)$ signature with an $\text{SO}(6, 6)$ Lorentz group embedded in the larger $E_{7(7)}$ symmetry. This example corresponds to choosing the final row of Table 1, given by $\mathfrak{F}^{0s} = \mathfrak{F}(\mathfrak{J}_3^{0s})$, the FTS defined over the Jordan algebra of split-octonionic 3×3 Hermitian matrices. Here $F(\tau, \sigma) \in \mathfrak{F}^{0s}$ transforms as the irreducible fundamental **56** of $\text{Aut}(\mathfrak{F}^{0s}) = E_{7(7)}$ and the quartic norm is given by Cartan's quartic invariant $\Delta(F) = I_4(F)$, defining an $E_{7(7)}$ -invariant generalisation of the Nambu-Goto action,

$$S_{E_{7(7)}} = \int d^2\xi \sqrt{|I_4(F)|} = \frac{1}{2} \int d\tau d\sigma \{\tilde{F}, F\}. \quad (3.25)$$

As before we can write an equivalent Polyakov-type action (3.20) by using the “non-critical” world-sheet \tilde{F} -duality defined by (3.19), where in this case $\mathcal{S}(\Phi) = \Omega\mathcal{M}(\Phi)$ is a function of 70 auxiliary scalar fields parametrising the coset $E_{7(7)}/\text{SU}(8)$.

As above, by substituting the equations of motion for the 70 auxiliary scalar fields Φ back into S_{pot} (3.20), one obtains $S_{E_{7(7)}}$ (3.25).

When $\text{Aut}(\mathfrak{F})$ is simple, as in this case, what we had previously treated as the world-sheet and spacetime indices are unified in the irreducible representation carried by \mathfrak{F} . To facilitate a string action interpretation a covariant split into world-sheet time/space derivatives must be made. Such a split is not necessarily unique, however, drawing on the conventional Nambu-Goto construction, a natural choice is given by the maximal embedding,

$$E_{7(7)} \supset \text{SL}(2, \mathbb{R}) \times \text{SO}(6, 6) \quad (3.26)$$

under which,

$$\mathbf{56} \rightarrow (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32}) \quad (3.27)$$

so that

$$F = (F_\alpha^a, \lambda^A) \quad \text{where} \quad \alpha = 1, 2 \quad a = 1, \dots, 12 \quad A = 1, \dots, 32. \quad (3.28)$$

The $(\mathbf{2}, \mathbf{12})$ F_α^a admits the usual interpretation as the world-sheet derivatives of the target space embedding coordinates $F_\alpha^a = \partial_\alpha X^a$ for a string propagating in a spacetime with $(6, 6)$ signature. On the other hand the $(\mathbf{1}, \mathbf{32})$ λ^A is a singlet under the $\text{SL}(2, \mathbb{R})$ and a spacetime Weyl spinor. Associating the $\text{SL}(2, \mathbb{R})$ indices with world-sheet derivatives, as we have done, implies λ^A is an auxiliary field.

The $\text{SL}(2, \mathbb{R})$ representations only determine the transformation properties of the fields under the subgroup of global world-sheet diffeomorphisms. We still need to specify their transformation rules under local world-sheet reparametrisations. Identifying F_α^a with $\partial_\alpha X^a$ fixes it as a world-sheet 1-form as usual. This, coupled with the requirement of world-sheet diffeomorphism invariance, implies that λ^A transforms as a world-sheet scalar density of weight (-1) . This follows from the decomposition of $I_4(F)$ under $\text{SL}(2, \mathbb{R}) \times \text{SO}(6, 6)$ which splits into three terms

$$I_4(F) = \det(F_\alpha^a F_\beta^b \eta_{ab}) + \alpha \epsilon^{\alpha\beta} F_{\alpha a} F_{\beta b} [\Gamma^{ab}]_{AB} \lambda^A \lambda^B + \beta [\Gamma^{ab}]_{AB} \lambda^A \lambda^B [\Gamma_{ab}]_{A'B'} \lambda^{A'} \lambda^{B'}. \quad (3.29)$$

If $\sqrt{|I_4(F)|}$ is to transform homogeneously as a weight (-2) scalar density as required for diffeomorphism invariance, then each summand in (3.29) must individually transform as a world-sheet scalar density of weight (-4) . This is true if and only if λ^A is a scalar density of weight (-1) , as claimed. Note that, under the transformations generated by the rank-4 symmetric para-quaternionic 64-dimensional coset algebra $\mathfrak{e}_{7(7)} \ominus (\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(6, 6))$, the fields F_α^a and λ^A mix together; this is consistent with the global $E_{7(7)}$ symmetry, which also fixes the real parameters α and β of (3.29).

It is here worth remarking that the Jordan algebras and FTS considered here have previously appeared in physics literature as the basis of generalised spacetimes [60], but from a rather different perspective.

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